UDC 536.2

ON A CLASS OF SOLUTIONS OF STEFAN'S FIRST BOUNDARY VALUE PROBLEM IN AN INFINITE SPACE FOR PLANE, AXIAL, AND SPHERICAL SYMMETRY

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Obtained are exact solutions of Stefan's first boundary value problem in a form suitable for direct use. Many other known solutions are particular cases of the class examined.

Successful solution of a whole range of current problems is closely linked with the question of finding the temperature or concentration field of a diffusing substance in the vicinity of a moving interface at which evolution or absorption of heat (or matter) takes place, the motion of the interface being itself determined by the unknown field. Most such problems lead to a nonlinear Stefan boundary value problem, the analysis and solution of which, even in the simplest cases, present serious mathematical difficulty, and frequently demand the application of numerical methods and computers [1-4].

At the same time, in the investigation of a series of nonstationary problems of heat conduction and diffusion by the methods of dimensional analysis, the heat conduction equation can often be reduced to a certain ordinary differential equation, whose solution presents no practical difficulty [5, 6]. E. M. Shakhov [7], applying a similar method to the solution of the Stefan problem, obtained certain self-similar solutions of it in an infinite space for plane, axial, and spherical symmetry.

In this paper the system of heat conduction equations is reduced to ordinary equations by a somewhat modified method.

Plane Problem in an Infinite and Semi-Infinite Space

We will examine the temperature field described by the equations

$$\frac{du_i}{dt} = a_i^2 \frac{\partial^2 u_i}{\partial z^2}, \quad i = 1,\ 2, \tag{1}$$

where $u_{1(t,z)}$ is defined in the region $z \geqslant \zeta_{(t)}$, and $u_{2(t,z)}$ in the region $z \leqslant \zeta_{(t)}$. Here, $\zeta_{(t)}$ is the coordinate of the moving interface, the motion of which is given by Stefan's calorimetric condition

$$\sigma \frac{d\zeta}{dt} = \lambda_1 \frac{du_1}{\partial z}\bigg|_{z=\zeta} - \lambda_2 \frac{\partial u_2}{\partial z}\bigg|_{z=\zeta}, \tag{2}$$

where σ, λ_i are constants, the significance of which depends on the specific problem examined.

On the left side of Eqs. (1) is the total derivative, which emphasizes the fact that the functions $u_{1(t,z)}$, $u_{2(t,z)}$ may depend on t both explicitly and parametrically, through the interface coordinate $\zeta_{(t)}$. We assume that $u_{i(t,z)}$ does not depend explicitly on t; it is well known that this is true of very many actual processes in the steady-state phase. With the object of more fully illustrating the method employed, we will examine the given problem in detail.

Developing du_i/dt in (1), we obtain

$$\frac{du_i}{\partial \zeta}\,\frac{d\zeta}{dt} = a_i^2 \frac{\partial^2 u_i}{\partial z^2}. \tag{3}$$

In a space not possessing an effective dimension, the functions u_i can depend only on the quantities σ, a_i^2, λ_i, z, $\zeta_{(t)}$ and on the quantities entering into the boundary conditions. It is easy to show that in the case of the first boundary value problem, i.e., for

$$\lim_{z\to\infty} u_{1(z,\zeta)} = T_1, \quad \lim_{z\to-\infty} u_{2(z,\zeta)} = T_2, \tag{4}$$

$$u_{1(\zeta,\zeta)} = T_1^*, \quad u_{2(\zeta,\zeta)} = T_2^*, \tag{5}$$

where T_i and T_i^* are independent of time, the only independent dimensionless variable is $x = z/\zeta_{(t)}$. Considering $u_i = u_{i(x)}$, we can reduce the system of equations (3) to a system of ordinary differential equations

$$-z\frac{d\zeta}{dt}\,\frac{du_i}{dx}=a_i^2\,\frac{d^2u_i}{dx^2}\,. \tag{6}$$

From dimensional considerations it is also obvious that

$$d\zeta/dt=\beta^2/2\zeta, \tag{7}$$

where β^2 is some constant, having the dimension of thermal diffusivity. Relationship (7) and

$$\zeta=\sqrt{\zeta_0^2+\beta^2 t}\,, \tag{8}$$

which follows from it, are necessary conditions for the self-similarity of the process. Solving (6), with account for (4), (5), (7), we obtain

$$\begin{aligned} u_{1(x)} &= T_1^*+(T_1-T_1^*)\,\frac{\operatorname{erf} k_1x-\operatorname{erf} k_1}{1-\operatorname{erf} k_1}\,,\\ u_{2(x)} &= T_2^*+(T_2-T_2^*)\,\frac{\operatorname{erf} k_2-\operatorname{erf} k_2x}{1+\operatorname{erf} k_2}\,, \end{aligned} \tag{9}$$

where $k_i = \beta_i/2a_i$.

Substituting $x = z/\zeta_{(t)}$ from (8) in (9) and setting $t = 0$, we obtain the possible class of initial conditions corresponding to the self-similar process of heated conduction reflected in (9):

$$u_{i(x,)}=\alpha_i+\beta_i\operatorname{erf}\gamma_i\,x_0. \tag{10}$$

The particular case of (9) with $T_1^* = T_2^*$, $\gamma_i = \infty$ is the known solution of [8]. It is easily seen that by appropriate choice of the quantities entering into (9), very different initial conditions can be satisfied.

The constant β^2 can be easily determined from condition (2).

In (5) the temperature was assumed capable of taking different values on different sides of the interface. Boundary conditions of this type can be useful in the approximate description of phase transitions in the temperature spectrum [5, 9]. Indeed, in considering, for example, the freezing of soil, we assume that the free water in the soil freezes at $z = z_{1(t)}$, where the temperature $T_1^* = 0$, while final freezing of the bound water takes place only at a certain negative temperature T_2^* corresponding to the level $z = z_{2(t)}$. Correct formulation of the problem requires examination of the heat conduction in the layer between $z_{1(t)}$ and $z_{2(t)}$ with the thermophysical characteristics of the medium depending on z [9]. However, since the thickness of this layer Δz is usually small in comparison with the scale factors of the process, it can be neglected, by choosing $\zeta_{(t)}$ anywhere in the interval $[z_{1(t)}, z_{2(t)}]$ and assuming boundary conditions in the form (5).

The concept of a self-similar process can be generalized by introducing the natural idea of a "quasi self-similar" process, i.e., a process described by functions of the form

$$u_i=\varphi_{i(z)}+f_{i(x)}, \tag{11}$$

where the f_i are solutions of Eqs. (6), and φ_i are steady-state solutions of Eqs. (1).

Retaining in (11) only the independent arbitrary constants, we obtain in general form

$$u_{i(z,\zeta)}=C_iz+D_i+B_i\operatorname{erfc} k_i\,z/\zeta_{(t)}. \tag{12}$$

Functions (12) describing the quasi self-similar process are easy to obtain, if we require that the right side of the calorimetric condition 2 with $u_{i(z,\zeta)}$ from (12) actually take the form (7). We obtain the condition

$$\lambda_1C_1=\lambda_2C_2. \tag{13}$$

As before, the constant β^2 is determined from (2).

Thus, instead of the four arbitrary constants appearing in the solution of (6) and typical of a self-similar process, we have five independent constants, which can be varied. Expressions (12) allow us to satisfy certain new types of boundary and initial conditions. We will demonstrate this on the example of a half-space $z \geq 0$. First of all, we note that a half-space is also without a characteristic dimension, so that all the derived formulas still hold true. For example, if

$$u_{2(0,\zeta)}=T_2 \tag{14}$$

we have $D_2 = T_2 - B_2$. With $C_i = 0$ and conditions (4), (5), functions (12) give an analog of the solution of [10]. If $C_i \neq 0$ and (14) is satisfied, then (12), with account for (13), gives a solution corresponding to the following initial temperature distribution:

$$u_{1(z,\zeta_0)} = \frac{\lambda_2}{\lambda_1} Cz + D_1 + B_1 \operatorname{erfc} k_1 \frac{z}{\zeta_0},$$
$$u_{2(z,\zeta_0)} = Cz + T_2 - B_2 \operatorname{erf} k_2 \frac{z}{\zeta_0}. \tag{15}$$

In this case the temperature at the moving surface $z = \zeta_{(t)}$:

$$u_{1(\zeta,\zeta)} = \frac{\lambda_2}{\lambda_1} C \sqrt{\zeta_0^2 + \beta^2 t} + D_1 + B_1 \operatorname{erfc} k_1,$$
$$u_{2(\zeta,\zeta)} = C \sqrt{\zeta_0^2 + \beta^2 t} + T_2 - \operatorname{erf} k_2, \tag{16}$$

i.e., depends on time. In particular, at $\zeta_0 = 0$ we have the solution for a linear initial temperature distribution in a semi-infinite medium. A similar problem was solved by D. V. Redozubov [11] for the case when $u_{2(\zeta,\zeta)}$ is a constant and $u_{1(\zeta,\zeta)}$ a variable. So far, we have considered the motion of an interface in the positive direction of the z axis. The opposite case is completely similar to that examined, and relations of the same type apply.

Axial Symmetry

Reasoning entirely analogously, we obtain in place of (6), (7), and (8),

$$\frac{d^2 f_i}{dx^2} + \frac{df_i}{dx}\left(\frac{1}{x} + 2k_i^2 x\right) = 0, \tag{17}$$

where $f_{1(x)}$ is defined at $x \geq 1$, $f_{2(x)}$ at $x \leq 1$, and

$$dR/dt = \beta^2/2R, \quad R = \sqrt{R_0^2 + \beta^2 t}. \tag{18}$$

In this case the quasi self-similar solutions have the form

$$u_{i(r,R)} = C_i \ln r + D_i + B_i \operatorname{Ei}\left(-k_i^2 \frac{r^2}{R^2}\right); \tag{19}$$

in contrast with the plane problem the calorimetric condition does not impose any limitations on the form of the arbitrary constants in (19). With $C_i = 0$ we obtain the self-similar solution of E. M. Shakhov [7]. It is easy to see how in this case $u_{2(r,R)}$ has a logarithmic singularity at the point $r = 0$, which corresponds to Shakhov's case of a point heat source (or sink) of constant intensity.

By varying the constants in (19), we can obtain solutions for different initial and boundary conditions. The only difference is that for the axisymmetric problem there are six such constants, and not five. For example, the condition of constant temperature at the interface (cylinder of radius $R_{(t)}$) will be $C_i = 0$. The condition for absence of a singular point $r = 0$ (no-source condition) is written thus:

$$2C_2 + B_2 = 0. \tag{20}$$

The latter is simple to obtain, using the representation of an integral exponential function

$$\operatorname{Ei}(-x) = C + \ln x + \int_0^x \frac{\exp(-\tau) - 1}{\tau} d\tau,$$

where $C = 0.5772\ldots$ is Euler's constant.

We shall examine the quasi self-similar processes satisfying (20) in more detail. First of all, with $C_2 \equiv 0$ and

$$u_{2(0,R)} = T_2, \quad u_{1(R,R)} = T_1^*, \quad \lim_{r\to\infty} u_{1(r,R)} = T_1, \tag{21}$$

we have the solution to the problem examined by D. E. Temkin in connection with the question of melting of a cylinder [12]. Consequently, in spite of the assertion of E. M. Shakhov that his solution is the only axisymmetric solution of the Stefan problem for self-similar processes [7], the solution

$$u_2 = T_2, \quad u_{1(r,R)} = T_1 + (T_1^* - T_1)\,\mathrm{Ei}\,(-k_1^2 r^2/R^2)/\mathrm{Ei}\,(-k_1^2), \tag{22}$$

describing, for example, the melting and crystallization of an infinite cylinder on the assumption that the thermal diffusivity of the internal region ($r \le R_{(t)}$) is much higher than that of the external zone, is also typically self-similar. All other solutions (19) not having a singular point at $r = 0$ lead to a variable value of the temperature on the inside of the heat front:

$$\begin{aligned} u_{1(r,R)} &= D_1 + C_1 \ln r + B_1\,\mathrm{Ei}\,(-k_1^2 r^2/R^2), \\ u_{2(r,R)} &= D_2 + B_2\,[\mathrm{Ei}\,(-k_2^2 r^2/R^2) - 2\ln r]. \end{aligned} \tag{23}$$

In particular, with $C_1 = 0$, Eqs. (23) describe the thermal processes in a medium, whose temperature at infinity is equal to D_1.

So far, we have examined the problem of an expanding interface. Converging motion, directed toward the axis of symmetry, is conveniently examined by introducing the new variable $y = 1/x = R/r$. Analysis shows that in this case the solution, as before, has the form (19). Physically this is easy to understand, considering that the corresponding processes (melting – crystallization, vaporization – condensation) are reversible.

As before, the constant β^2 must be determined from condition (2).

Spherical Symmetry

Repeating the calculations, we have in this case in place of (12) or (19)

$$u_{i(r,R)} = D_i + \frac{C_i}{r} + B_i \frac{\mathrm{ierfc}\,k_i x}{x}, \quad x = \frac{r}{R_{(t)}}, \tag{24}$$

where the function

$$\mathrm{ierfc}\,z = \frac{1}{\sqrt{\pi}} \exp(-z^2) - z\,\mathrm{erfc}\,z$$

was introduced by D. V. Redozubov [11].

The conditions of quasi self-similarity, corresponding to condition (13) for the plane problem, is written in the form

$$\lambda_1 C_1 = \lambda_2 C_2. \tag{25}$$

The temperature at the moving interface will not depend on time if $C_i = 0$. The condition of absence of singularities assumes the form

$$C_2 + B_2 \frac{R}{\sqrt{\pi}} = 0 \quad \text{or} \quad C_2 = B_2 = 0. \tag{26}$$

The self-similar solution of E. M. Shakhov is a special case of (24) with $C_i = 0$. There is also a second self-similar solution, corresponding to constant temperature inside the sphere $R_{(t)}$. This solution can be used for a mathematical description of the vaporization of droplets. The general spherically-symmetric problem does not differ in principle from those already examined; therefore all that has been said about the plane or axially symmetric problem applies in this case too.

NOTATION

β^2 – a certain constant, having the dimension of thermal diffusivity; erf x – error integral; α_i, β_i, and γ_i – constants which may be varied by changing T_i, T_i^*, and ζ_0; $x = z/\zeta$.

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